# Compound Interest and Annuities

Finite Math

26 February 2019

# Compound Interest

As before, we can use these compound interest models to figure out how much we should invest now to achieve a desired future value.

# Compound Interest

As before, we can use these compound interest models to figure out how much we should invest now to achieve a desired future value.

We can also look to see how long something will take to mature given the principal, the growth rate, and the desired future value. The power rule for logarithms comes especially in handy here: $\log_b M^p = p \log_b M$.

# Compound Interest

As before, we can use these compound interest models to figure out how much we should invest now to achieve a desired future value.

We can also look to see how long something will take to mature given the principal, the growth rate, and the desired future value. The power rule for logarithms comes especially in handy here: $\log_b M^p = p \log_b M$.

**Example**

*How long will it take* $\$10,000$ *to grow to* $\$25,000$ *if it is invested at* $8\%$ *compounded quarterly?*

# Now You Try It!

### Example

*How long will it take* $\$2,000$ *to grow to* $\$6,000$ *if it is invested at (a)* $5\%$ *compounded daily? (b)* $6\%$ *compounded continuously? (in part (b), round to* $3$ *decimal places)*

# Now You Try It!

## Example

*How long will it take* $\$2,000$ *to grow to* $\$6,000$ *if it is invested at (a)* $5\%$ *compounded daily? (b)* $6\%$ *compounded continuously? (in part (b), round to* $3$ *decimal places)*

## Solution

(a) $8,021$ *days (about* $21.975$ *years)*

(b) $18.310$ *years*

# Compound Interest

We can also look to figure out the desired interest rate if we know the present value, the length of time, and the desired future value.

# Compound Interest

We can also look to figure out the desired interest rate if we know the present value, the length of time, and the desired future value.

## Example

*The Russell Index tracks the average performance of various groups of stocks. On average, a* $\$10,000$ *investment in mid-cap growth funds over a* $10$*-year period would have grown to* $\$63,000$*. What annual nominal rate would produce the same growth if interest were compounded (a) annually, (b) continuously? Express answers as a percentage, rounded to three decimal places.*

# Now You Try It!

## Example

*A promissory note will pay* $\$50,000$ *at maturity* $6$ *years from now. If you pay* $\$28,000$ *for the note now, what rate would you earn if interest were compounded (a) quarterly, (b) continuously?*

# Now You Try It!

## Example

*A promissory note will pay* $\$50,000$ *at maturity* $6$ *years from now. If you pay* $\$28,000$ *for the note now, what rate would you earn if interest were compounded (a) quarterly, (b) continuously?*

## Solution

(a) 9.78%

(b) 9.66%

# Annuities

At this point, we have only discussed investments where there was one initial deposit and a final payoff. But what if you make regular equal payments into an account?

# Annuities

At this point, we have only discussed investments where there was one initial deposit and a final payoff. But what if you make regular equal payments into an account? An *annuity* is a sequence of equal periodic payments. If payments are made at the end of each time interval, then the annuity if called an *ordinary annuity*. Our goal will be to find the future value of an annuity.

# Future Value of an Annuity

## Example

*Suppose you decide to deposit* $100 *every* 6 *months into a savings account which pays* 6% *compounded semiannually. If you make* 6 *deposits, one at the end of each interest payment period over the course of* 3 *years, how much money will be in the account after the last deposit is made?*

# Solution

We can visualize the value of each of those \$100 deposits in a table.

## Solution

We can visualize the value of each of those \$100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|

# Solution

We can visualize the value of each of those $100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| $100 | | | |

# Solution

We can visualize the value of each of those $100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| $100 | 1 | | |

# Solution

We can visualize the value of each of those $100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| $100 | 1 | 5 | |

# Solution

We can visualize the value of each of those $100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| $100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |

# Solution

We can visualize the value of each of those $100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| \$100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| \$100 | | | |

# Solution

We can visualize the value of each of those $100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| \$100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| \$100 | 2 | | |

# Solution

We can visualize the value of each of those \$100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| \$100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| \$100 | 2 | 4 | |

# Solution

We can visualize the value of each of those \$100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| \$100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| \$100 | 2 | 4 | $\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$ |

# Solution

We can visualize the value of each of those \$100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| \$100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| \$100 | 2 | 4 | $\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$ |
| \$100 | 3 | 3 | $\$100\left(1+\frac{0.06}{2}\right)^3 = \$100(1.03)^3$ |

# Solution

We can visualize the value of each of those \$100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| \$100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| \$100 | 2 | 4 | $\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$ |
| \$100 | 3 | 3 | $\$100\left(1+\frac{0.06}{2}\right)^3 = \$100(1.03)^3$ |
| \$100 | 4 | 2 | $\$100\left(1+\frac{0.06}{2}\right)^2 = \$100(1.03)^2$ |

# Solution

We can visualize the value of each of those $100 deposits in a table.

| **Deposit** | **Term** | **# of times Compounded** | **Future Value** |
|---|---|---|---|
| $100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| $100 | 2 | 4 | $\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$ |
| $100 | 3 | 3 | $\$100\left(1+\frac{0.06}{2}\right)^3 = \$100(1.03)^3$ |
| $100 | 4 | 2 | $\$100\left(1+\frac{0.06}{2}\right)^2 = \$100(1.03)^2$ |
| $100 | 5 | 1 | $\$100\left(1+\frac{0.06}{2}\right)^1 = \$100(1.03)$ |

# Solution

We can visualize the value of each of those \$100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| \$100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| \$100 | 2 | 4 | $\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$ |
| \$100 | 3 | 3 | $\$100\left(1+\frac{0.06}{2}\right)^3 = \$100(1.03)^3$ |
| \$100 | 4 | 2 | $\$100\left(1+\frac{0.06}{2}\right)^2 = \$100(1.03)^2$ |
| \$100 | 5 | 1 | $\$100\left(1+\frac{0.06}{2}\right)^1 = \$100(1.03)$ |
| \$100 | 6 | 0 | $\$100\left(1+\frac{0.06}{2}\right)^0 = \$100$ |

# Solution

We can visualize the value of each of those \$100 deposits in a table.

| Deposit | Term | # of times Compounded | Future Value |
|---|---|---|---|
| \$100 | 1 | 5 | $\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$ |
| \$100 | 2 | 4 | $\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$ |
| \$100 | 3 | 3 | $\$100\left(1+\frac{0.06}{2}\right)^3 = \$100(1.03)^3$ |
| \$100 | 4 | 2 | $\$100\left(1+\frac{0.06}{2}\right)^2 = \$100(1.03)^2$ |
| \$100 | 5 | 1 | $\$100\left(1+\frac{0.06}{2}\right)^1 = \$100(1.03)$ |
| \$100 | 6 | 0 | $\$100\left(1+\frac{0.06}{2}\right)^0 = \$100$ |

So adding up the future values of all these will give us the amount of money in the account

$$\begin{aligned} B &= \$100(1.03)^5 + \$100(1.03)^4 + \$100(1.03)^3 \\ &\quad + \$100(1.03)^2 + \$100(1.03) + \$100 = \$646.84 \end{aligned}$$

# Future Value

**Definition (Future Value of an Ordinary Annuity)**

$$FV = PMT \frac{\left(1 + \frac{r}{m}\right)^n - 1}{r/m}$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^{n}-1}{r/m}$$

*where*

$$FV$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT \frac{\left(1 + \frac{r}{m}\right)^n - 1}{r/m}$$

*where*

$$FV = \text{future value}$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^n - 1}{r/m}$$

*where*

$$\begin{aligned} FV &= \text{future value} \\ PMT & \end{aligned}$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^n - 1}{r/m}$$

*where*

$$\begin{aligned} FV &= \text{future value} \\ PMT &= \text{periodic payment} \end{aligned}$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^{n}-1}{r/m}$$

*where*

$$\begin{aligned} FV &= \text{future value} \\ PMT &= \text{periodic payment} \\ m & \end{aligned}$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^n - 1}{r/m}$$

*where*

$$\begin{aligned} FV &= \textit{future value} \\ PMT &= \textit{periodic payment} \\ m &= \textit{frequency of payments} \end{aligned}$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^n - 1}{r/m}$$

*where*

$$\begin{aligned} FV &= \text{future value} \\ PMT &= \text{periodic payment} \\ m &= \text{frequency of payments} \\ n & \end{aligned}$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT \frac{\left(1+\frac{r}{m}\right)^n - 1}{r/m}$$

*where*

$$\begin{aligned} FV &= \textit{future value} \\ PMT &= \textit{periodic payment} \\ m &= \textit{frequency of payments} \\ n &= \textit{number of payments (periods)} \end{aligned}$$

# Future Value

## Definition (Future Value of an Ordinary Annuity)

$$FV = PMT\frac{\left(1+\frac{r}{m}\right)^n - 1}{r/m}$$

*where*

$$\begin{aligned} FV &= \textit{future value} \\ PMT &= \textit{periodic payment} \\ m &= \textit{frequency of payments} \\ n &= \textit{number of payments (periods)} \end{aligned}$$

*Note that the payments are made at the end of each period.*

# Future Value

**Example**

*What is the value of an annuity at the end of* $10$ *years if* $\$1,000$ *is deposited every* $3$ *months into an account earning* $8\%$ *compounded quarterly. How much of this value is interest?*

# Now You Try It!

## Example

*If* $\$1,000$ *is deposited at the end of each year for* $5$ *years into an ordinary annuity earning* $8.32\%$ *compounded annually, what will be the value of the annuity at the end of the* $5$ *years?*

# Now You Try It!

## Example

*If $\$1,000$ is deposited at the end of each year for $5$ years into an ordinary annuity earning $8.32\%$ compounded annually, what will be the value of the annuity at the end of the $5$ years?*

## Solution

$\$5,904.15$

# Sinking Funds

We can turn the annuities picture around and ask how much we would need to deposit into an account each period in order to get the desired final value.

# Sinking Funds

We can turn the annuities picture around and ask how much we would need to deposit into an account each period in order to get the desired final value.

**Example**

*New parents are trying to save for their child's college and want to save up* $\$80,000$ *in* $17$ *years. They have found an account that will pay* $8\%$ *interest compounded quarterly. How much will they have to deposit every quarter in order to have a value of* $\$80,000$*?*

# Now You Try It!

## Example

*A bond issue is approved for building a marina in a city. The city is required to make regular payments every* $3$ *months into a sinking fund paying* $5.4\%$ *compounded quarterly. At the end of* $10$ *years, the bond obligation will be retired with a cost of* $\$5,000,000$. *How much will the city have to pay each quarter?*

# Now You Try It!

## Example

*A bond issue is approved for building a marina in a city. The city is required to make regular payments every* $3$ *months into a sinking fund paying* $5.4\%$ *compounded quarterly. At the end of* $10$ *years, the bond obligation will be retired with a cost of* $\$5,000,000$. *How much will the city have to pay each quarter?*

## Solution

$\$95,094.67$